Talk Notes - Presheaves and Sheaves

For the talk given on Wednesday, 19.10.22 by A. Roth

Seminar on Sheaf Theory WiSe22/23

Contents

1 Sheaves and Presheaves

For the following, let X be a topological Space.

Definition 1. *A* presheaf *(of sets) F on X is a structure consisting of* sets of sections $F(U)$ *associated to each open subset $U \subset X$ and* restriction maps *between the respective sets of sections for all open $U, V \subset X$ with $V \subset U$, such that:*

(P1) For all open $U \subset X$, $F(U)$ is a set,

(P2) For all $V \subset U \subset X$, V, U open, the restriction map $\rho_V^U : F(U) \to F(V)$, satisfies

(a) $\rho_U^U = id_U$ for all open $U \subset X$

(b) $\rho_W^U = \rho_W^V \circ \rho_V^W$ for all open $W \subset V \subset U$

Building on this definition, the presheaf of an abelian group can be defined as follows:

Definition 2. *A* presheaf *(of abelian groups) is a presheaf F of sets such that*

(P1') For all open $U \subset X$, $F(U)$ has abelian group structure,

(P2') Every restriction map is a group homomorphism.

In a similar fashion, presheaves can be defined vor various different structures by requiring the sections to have the respective structure and the restriction maps to be morphisms in respect to the structure. With the existence of restriction maps, a presheaf guarantees the sections will behave nicely under restriction of the associated sets. A question that one might ask now is what happens on a larger scale. Can local sections that coincide on intersections be extended to a global section in a meaningful way? Are two everywhere locally equal sections also globally equal? The answer to both of these questions is no in general; if those two properties hold, however, the presheaf can be used in much broader contexts, leading to the following definition of a sheaf.

Definition 3. *A* Sheaf *(of sets) is a presheaf F (of sets) that satisfies*

(M) the Monopresheaf condition*: For any $U \subset X$ open, let $\mathcal{U} = \bigcup_{i \in I} U_i$ be an open covering of U. Further, let $s, s' \in F(U)$ such that $\rho^U_{U_i}(s) = \rho^U_{U_i}(s')$ for all $i \in I$. Then, $s' = s$.*

(G) the Glueing condition*: For any $U \subset X$ open, let $\mathcal{U} = \bigcup_{i \in I} U_i$ be an open covering of U. Now, let $(s_i)_{i \in I}$ with $s_i \in F(U_i)$ $\forall i \in I$ such that $\forall i, j \in I : \rho^{U_i}_{U_i \cap U_j}(s_i) = \rho^{U_j}_{U_i \cap U_j}(s_j)$.. Then there exists $s \in F(U)$, such that $rho^U_{U_i}(s) = s_i$ $\forall i \in I$.*

In words, the monopresheaf condition guarantees that two sections that are equal locally are already equal on the whole set; the glueing condition gives the opportunity to „glue“ together local sections to a global section, permitting to extend sections onto a bigger domain under certain circumstances. Between the presheaf conditions and the sheaf conditions, getting „bigger“ is regulated as well as getting „smaller“ . As for presheaves, sheaves can be defined accordingly for various different structures.

2 Examples, Part One

Now, it's time to see some basic examples of presheaves and sheaves. For each example, we will shortly check which of the conditions as defined above holds.

Example 1. *For a given set A, the* constant presheaf *A_X is defined by*

$$\begin{cases} A_X(U) := A & \text{for } U \subset X \text{ open} \\ \rho^U_V := id_A & \forall V \subset U \subset X \text{ open} \end{cases}$$

One immediatley sees that the constant presheaf indeed satisfies the presheaf conditions (P1) and (P2), as the name implies. Additionally, it satisfies the monopresheaf condition: As the restriction map is always the identity on A, so for any open U with open covering and $s, s' \in F(U)$, $\rho^U_{U_i}(s) = \rho^U_{U_i}(s')$ directly implies $s = s'$.

The constant presheaf is in general not a sheaf, however, as it doesn't satisfy the glueing condition.

Example 2. *Define the presheaf F_{const} of constant functions by*

$$\begin{cases} F_{const}(U) := \{f : U \to \mathbb{R} \text{ constant}\} & \text{for } U \subset X \text{ open} \\ \rho^U_V := f|_V & \forall V \subset U \subset X \text{ open} \end{cases}$$

The presheaf of constant functions is quite similar to the constant presheaf. Once again, it is easy to see that (P1) and (P2) hold. (M) follows from the fact that all the considered functions are constantz. Once again, the glueing condition fails, and here it's easy to see why: Consider two open, disjoint sets $U_1, U_2 \subset X$ and two sections $f_1 \in F_{const}(U_1)$, $f_2 \in F_{const}(U_2)$ taking on different values. U_1 and U_2 clearly form an open covering of an open set $U = U_1 \cup U_2 \subset X$, however, their intersection is empty. As such, the sections coincide when restricted to $U_1 \cap U_2$: $f_1|_\emptyset = f_2|_\emptyset$. As f_1 and f_2 take on different values, there is no constant function on $U_1 \cup U_2$ that restricts to both at the same time, and thus, no section that satisfies the glueing condition.

Note that this is different if instead of constant functions we choose *locally* constant functions, i.e. functions that are constant on each connected component. Indeed, the locally constant presheaf is a sheaf! This is because it is the „Sheafification“ of the sheaf of constant functions, which will be covered in a later talk.

Next up are more function sheaves that find common use:

Example 3. *The sheaf of* r-times differentiable functions *is given by*

$$\begin{cases} C^r(U) := \{f : U \to \mathbb{R} | f \text{ is r-times continuously differentiable}\} & \text{for } U \subset X \text{ open} \\ \rho^U_V \cdot = f|_V & \forall V \subset U \subset X \text{ open} \end{cases}$$

Similarly, the sheaves C^∞ and C^ω can be defined, with $C^\omega(U)$ denoting holomorphic functions on U.

Again, (P1) and (P2) are easy to verify, as well as (M), considering the functions are differentiable. The glueing condition can be checked by using the fact that r-times continuously differentiable functions have this property on a small neighborhood of each point.

It is to note here that the sheaves C^∞ and C^ω behave quite differently. Also, one can see where ρ got the name „restriction map“ . To see where the name of the sections comes from, let's consider the last example, the sheaf of sections.

Example 4. *Let E be a topological space, $p : E \to X$ continuous. The sheaf of sections ist defined by*

$$\begin{cases} F(U) := \{\sigma : U \to E \text{ cont.} | p \circ \sigma = id_U\} & \text{for } U \subset X \text{ open} \\ \rho^U_V : F(U) \to F(V), \sigma \mapsto \sigma|_v & \forall V \subset U \subset X \text{ open} \end{cases}$$

Here, (P1) and (P2) are easy to see. (M) is a consequence of the fact that $p \circ \sigma = id_U$, σ has to be an injection and therefore the image of a point is unique. (G) follows by the fact that an open covering is considered, so the function is smooth on the glueing edges.

THe sections in this example are literally sections as known from topology.

3 Direct Limits, Stalks and Germs

After considering some examples, we now want to lay the basis for later constructions considering sheaves and introduce the terminology of germ and stalk. For this, however, we first have to define what directed systems and their direct limits are. Basically, the idea of a directed system is to order structures (in our case sets) in order of them getting „smaller“ with the direct limit being the „smallest“ such element of a directed system.

Definition 4. *A* directed set Λ *is a set with a preorder* $\leq$ *that also satisfies: for all* $\alpha, \beta \in \Lambda$ *there exists* $\gamma \in \Lambda$ *such that* $\alpha \leq \gamma$ *and* $\beta \leq \gamma$*. A* directed system *(of Sets) is a family of sets* $(U_\alpha)_{\alpha \in \Lambda}$ *together with a map* $\rho_{\alpha\beta} : U_\alpha \to U_\beta$ *for each pair* $\alpha, \beta \in \Lambda$ *with* $\alpha \leq \beta$ *that satisfies the following conditions:*

(a) $\rho_{\alpha\alpha} = id_{U_\alpha} \ \forall \alpha \in \Lambda$

(b) $\rho_{\alpha\gamma} = \rho_{\beta\gamma} \circ \rho_{\alpha\beta} \ \forall \alpha.\beta, \gamma \in \Lambda$ *with* $\alpha \leq \beta \leq \gamma$.

The similarity of the ρ in the directed system and te restriction maps of presheaves is striking and, in the end, is exactly what we will use for our causes.

Definition 5. *A* target *of a directed system (of sets)*$(U_\alpha)_{\alpha \in \Lambda}$ *is a set* V *and a collection of maps* $\sigma_\alpha : U_\alpha \to V$ *such that for all* $\alpha, \beta \in \Lambda, \alpha \leq \beta$ *the* $\sigma_\alpha, \sigma_{beta}$ *are compatible with the* $\rho_{\alpha\beta}$ *in the sense that* $\sigma_{alpha} = \sigma_{beta} \circ \rho_{\alpha\beta}$*. A* direct limit *for the system is a target* U *with maps* $\tau_\alpha : U_\alpha \to U$ *that satisfies the following universal property: for any target* V *as above, there exists a unique map* $f : U \to V$ *such that* $\sigma_\alpha = f \circ \tau_\alpha \ \forall \alpha \in \Lambda$.

Every directed system has a direct limit, and two direct limits of the same directed system are naturally isomorphic, so it makes sense to think of „the“ direct limit of a directed system. As this is not the main emphasis of the talk, please refer to Chapter 1.3 of [Tennison].

As before, these definitions can similarly be formulated for other structures by requiring the elements of the directed system having the respective structures and the maps $\rho_{\alpha\beta}$ to be morphisms of this structure.

With the concept of direct limits, we can now talk about stalks and germs by considering a system of progressively smaller subsets of X containing a specific Element. Note that te following definition s formulated for presheaves; in fact stalks are one key component for the sheafification of a presheaf.

Definition 6. *Let* F *be a presheaf. For fixed* $x \in X$ *consider a directed system of* $F(U)$ *with* $x \in U$*, using the restriction maps of* F*. The* stalk F_x *of* F *at* x *then is the direct limit* $lim_{\overrightarrow{U \ni x}}$ *with maps* $F(U) \to F_x, s \mapsto s_x$ *for* $x \in U$*,* U *open. The members of a stalk are called* Germs *of sections.*

Conceptually, the stalks and germs are the extension from presheaves over sets $U \subset X$ to include points in X and the smallest object associated with the presheaf and compatible with the restriction maps. They will in later talks find use as a tool to handle sheaves where sections are „too big“ .

4 Examples, Part Two

Finally, let's take a look at two examples of stalks, the first one being the stalks of a constant presheaf as seen in 1. It follows directly from the definition of that presheaf.

Example 5. *The stalk of a constant presheaf of a set* A *as seen in Example 1 is given by*

$$A_{X,x} = A \ \ \forall x \in X.$$

The second example are stalks of the presheaf F_{const} at x. It consists of equivalence classes of constant functions on a small neighbourhood of x with two functions being equivalent if they coincide restricted to a neighbourhood of x.

Example 6. *The stalk of the presheaf of constant functions is given by*

$$\{f : U_x \to \mathbb{R} |\ f \text{ constant}\}_{/\sim}$$

with $f \sim g$ *iff* $f|_{V_x} = g|_{V_x}$ *for a small* $V_x \subset U_x$.

Note that this is also the stalk of the sheaf of locally constant functions – stalks cannot be uniquely identified with sheaves! With this, the groundwork for the upcoming talks is done.

5 Sources

[Tennison] B. R. Tennison, *Sheaf Theory.* London Mathematical Society Lecture Notes Series, No. 20. Cambridge University Press, Cambridge, Endgland-New York-Melbourne, 1975

6 Remarks

This section contains a remark I made in the talk concerning the naming of sheaves. A sheaf (*Garbe*) is originally a bundle of wheat. If one imagines such a bundle (see below), the notion of sections (*Schnitte*) and stalks (*Halme*) seems obvious. Note, however, that the notion of (mathematical) germs does not intuitively correspond to the germs of wheat: The (mathematical) germs of a sheaf can sit at any point on the stalk and give rise to sections of the sheaf, rather than the stalks!